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Exact description of black holes on branes II: comparison with BTZ black holes and black strings

Roberto Emparan

Departamento de Física Teórica, Universidad del País Vasco
Apdo. 644, E-48080 Bilbao, Spain
E-mail: wtpemgar@lg.ehu.es

Gary T. Horowitz

Physics Department, University of California
Santa Barbara, CA 93106 USA
E-mail: gary@cosmic.physics.ucsb.edu

Robert C. Myers

Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106 USA, and
Department of Physics, McGill University
Montréal, QC, H3A 2T8, Canada
E-mail: rcm@hep.physics.mcgill.ca

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Abstract: We extend our recent discussion of four-dimensional black holes bound to a two-brane to include a negative cosmological constant on the brane. We find that for large masses, the solutions are precisely BTZ black holes on the brane, and BTZ 'black strings' in the bulk. For smaller masses, there are localized black holes which look like BTZ with corrections that fall off exponentially. We compute when the maximum entropy configuration changes from the black string to the black hole. We also present exact solutions describing rotating black holes on two-branes which are either asymptotically flat or asymptotically AdS_3. The mass and angular momentum on the brane agree with that in the bulk.

Keywords: p-branes, Black Holes.

Contents

1. Introduction

In a recent paper [1] we constructed exact solutions describing four dimensional black holes bound to a two-brane in anti-de Sitter (AdS) space. This construction was motivated by the suggestion [2] that lower dimensional gravity is naturally induced on a brane (domain wall) placed in AdS spacetime [3]. The original discussion involved linearized perturbations of a three-brane in AdS_5, but there were questions raised [4] about the boundary conditions used near the AdS horizon. By considering a two-brane in AdS_4, we were able to investigate the viability of this scenario by studying the properties of exact black hole solutions.

We found that lower dimensional gravity was indeed reproduced on the brane at large distances, with no difficulty arising at the AdS horizon. (The horizon geometry was, in fact, unchanged.) Large black holes appeared as flattened pancakes with a much smaller extent off the brane than along the brane. These features will extend to the five dimensional case. However, there were some awkward features of our solutions. The effects of four-dimensional gravity were significant even at scales much larger than the bulk cosmological constant. This is perhaps not surprising given that $2+1$ gravity (without a cosmological constant on the brane) does not have any black hole solutions. Another unusual feature of $2+1$ gravity is that there is a maximum total mass. We found that as the mass approached this limit, the size of the black hole grew without bound. Thus one can have arbitrarily large black holes with finite total mass.

In this paper we extend our previous analysis to allow a negative cosmological constant in the $2+1$ gravity theory induced on the brane. This has the advantage that both of the unusual features of pure $2+1$ gravity mentioned above are avoided. When a negative cosmological constant is added, $2+1$ gravity does have black hole solutions [5], and there is no upper limit on the total mass. Thus one might hope that this is a more realistic model of the higher dimensional physics.

The solutions in [1] were obtained by starting with a metric describing an accelerating black hole in AdS_4. We then introduced the two-brane by cutting the spacetime along an appropriate surface and gluing it to a copy of itself. To add a cosmological constant on the brane we start with a slightly more general solution, but in this case we have to compactify the fourth direction by introducing two different branes. We will argue that this is a general feature of branes with negative curvature and is not special to AdS_4. Unlike previous discussions of compactifications with two branes [6], both of our branes have positive tension.

We find that large black holes on the brane agree exactly with the $2+1$ dimensional BTZ solution. However, these black holes are not localized near the brane. They extend across the fourth dimension and form a 'BTZ black string'. It is perhaps not surprising that when the size of the extra dimension is finite, large black holes on the brane correspond to black strings in the bulk. This is what one expects from a standard Kaluza-Klein compactification. For smaller masses, we have two different black hole solutions in which the horizon does not reach the second brane. In both cases, the solution looks like the BTZ black hole with correction terms that fall off exponentially in the proper distance. These terms are negligible outside the horizon for the solution with smaller area. Off the brane, it looks like the BTZ black string with its ends capped off. In other words, in this case, the horizon looks more like a cigar than a pancake. The solution with larger horizon area can have significant deviation from the BTZ solution outside the black hole.

Another motivation for considering a negative cosmological constant on the brane is the following. It has been argued in higher dimensions that black strings are unstable when their width shrinks to less than the AdS scale [7]. Since we now have exact solutions for both the black strings and localized black holes, we can examine this transition in detail. We find that indeed the maximum entropy configuration changes from the black string to the black hole when the minimum cross-sectional area is of order the AdS radius.

For black holes on asymptotically flat two-branes, it was shown [1] that the mass measured asymptotically on the brane agreed with a four dimensional thermodynamic mass obtained by integrating the relation $dM = TdS$ in the bulk. We show that the same is true for black holes on asymptotically AdS_3 branes.

We also construct solutions describing rotating black holes on a two-brane both with and without a cosmological constant on the brane. In the first case, we recover on the brane the metric of rotating BTZ black holes with some modifications. In the latter case, the metric on the brane is locally equivalent to the metric on the equatorial plane of the four-dimensional Kerr solution. We will see that the rotation of the black hole in the four-dimensional space is transferred to the brane in such a way that the angular momentum detected on the brane is precisely equal to the four-dimensional angular momentum.

2. The AdS C-metric and two-branes

We begin with the following solution to Einstein's equation with negative cosmological constant describing accelerating black holes in AdS_4 [8]:

$$ds^2 = \frac{1}{A^2(x-y)^2}\left[H(y)dt^2 - \frac{dy^2}{H(y)} + \frac{dx^2}{G(x)} + G(x)d\varphi^2\right], \tag{2.1}$$

where

$$H(y) = -\lambda + ky^2 - 2mAy^3\,, \tag{2.2}$$

$$G(x) = 1 + kx^2 - 2mAx^3 \tag{2.3}$$

and $k = +1, 0, -1$. The metric satisfies $R_{AB} = -(3/\ell_4^2)g_{AB}$, where

$$\ell_4 = \frac{1}{A\sqrt{\lambda+1}} \tag{2.4}$$

sets the scale for the bulk cosmological constant i.e. $\Lambda_4 = -3/\ell_4^2$. Note that from eq. (2.4), we require $\lambda \geq -1$. In the limit $\lambda \to -1$ (with $k = -1$), we recover the C-metric [9] describing a pair of black holes accelerating in an asymptotically flat spacetime. The general solution with $\lambda > -1$ is called the AdS C-metric. This describes black holes accelerating in AdS_4. The parameters m and A are related to the mass and acceleration of the black hole (at least for certain limits — see below).

Another interesting limit reduces the metric (2.1) to that of static four-dimensional black holes in AdS_4 [10]. This limit turns off the acceleration parameter A as follows: set $y = -1/(rA)$, $t = A\bar{t}$, $\lambda = -\Lambda_4/3A^2 > 0$, and take the limit $A \to 0$ while keeping $\bar{t}, r, x, \varphi$ and Λ_4 finite. In this way m gives the mass parameter of the static black hole solutions of Einstein's equations with a negative cosmological constant Λ_4. Recall that the horizons of the static $k = -1$ black holes are topological spheres and have finite area. In contrast, the $k = 0, +1$ black holes have horizons with the topology of R^2 and infinite area, The special case of (2.1) with $\lambda = 0$ and $k = -1$ was employed in [1]. Many of the descriptive comments made there still apply for the general metric (2.1). For example, due to the overall factor $(x-y)^{-2}$, the solution does not include points with $x = y$. Generically, these points correspond to the boundary of the asymptotically AdS geometry. Hence, we will restrict y to always be less than x. There is also a curvature singularity at $y = -\infty$. Clearly both t and φ are Killing coordinates. As the notation indicates φ is an angular coordinate and each zero of $G(x)$ corresponds to an axis for the rotation symmetry. Similarly t may be regarded as the time coordinate, and each zero of $H(y)$ corresponds to a Killing horizon for ∂_t. The smallest zero y_0 of $H(y)$ defines the black hole event horizon. Typically one finds that the accelerating black holes have horizons with the topology of R^2 and infinite area, for all k values.[1] Only for $k = -1$ and mA not 'too large', is the event horizon a sphere with finite area.

To orient ourselves with the metric (2.1), let us set $m = 0$. The geometry then has constant curvature and is locally the same as AdS_4. To cast the metric into a more familiar form define

$$r = \frac{\sqrt{y^2 + \lambda x^2}}{A(x-y)}\,, \qquad \rho = \sqrt{\frac{1 + kx^2}{y^2 + \lambda x^2}}\,. \tag{2.5}$$

Then (2.1) becomes

$$ds^2 = \frac{dr^2}{\frac{r^2}{\ell_4^2} - \lambda} + r^2\left[-(\lambda\rho^2 - k)dt^2 + \frac{d\rho^2}{\lambda\rho^2 - k} + \rho^2 d\varphi^2\right]. \tag{2.6}$$

These forms of the AdS_4 metric had been considered earlier in [11]. The metric in the brackets has constant Riemann curvature $\mathcal{R}^{\mu\nu}{}_{\rho\sigma} = -\lambda\,(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho)$. When $m = 0$, λ can be rescaled by rescaling the coordinates r and ρ, and so one may set $\lambda = 0$ or ± 1. These three choices correspond to three distinct ways of slicing AdS_4. Notice that $\lambda = 0,\ k = -1$ corresponds to the usual Poincaré invariant slices.

To introduce a brane into the spacetime, we need a surface whose extrinsic curvature is proportional to the intrinsic metric. Then we can cut the spacetime off at the surface, take two copies of one side, and glue them together. The resulting space will have a delta-function stress tensor along the surface which is proportional to the induced metric, i.e. a two-brane. The extrinsic curvature of a surface can be computed via

$$K_{\mu\nu} = \nabla_\mu\, n_\nu = \frac{1}{2} n^\sigma \partial_\sigma g_{\mu\nu}\,, \tag{2.7}$$

where μ, ν run over the components tangent to the surface, and n^σ is the unit outward pointing normal vector. (The last formula results in the special case that $\partial_\mu n^\sigma = 0$,

[1] Note that this infinite area is not a problem for the present analysis. As in [1], the infinite tail of the horizon will be cut out by the introduction of a two-brane and we will obtain a black hole of finite size.

which will apply in the cases of interest for this paper.) For example, a short calculation reveals that slices at constant r in eq. (2.6) satisfy $K_{\mu\nu} = r^{-1}\sqrt{r^2/\ell_4^2 - \lambda}\, g_{\mu\nu}$. Hence the space can be cut at any of these surfaces and glued to a copy of itself to construct domain walls (two-branes).

Since a surface $r = L$ has constant curvature, there is a cosmological constant $\Lambda_3 = -\lambda/L^2$ on the brane. Hence there are three distinct cases: (i) $\lambda = 0$, in which case the radial slices are flat. Here we may only choose $k = \pm 1$. The choice $k = -1$ yields global Minkowski coordinates with t timelike everywhere. (ii) $\lambda < 0$, in which case the radial slices have the geometry of three-dimensional de Sitter space, dS_3. With $k = -1$, t is timelike for small ρ and there are de Sitter horizons on the two-brane at $\rho = |\lambda|^{-1/2}$. The latter correspond to $y^2 = -\lambda$ in the C-metric coordinates of eq. (2.1). (iii) $\lambda > 0$, where locally the constant r slices have the geometry of AdS_3. The three possible values of k yield different parameterizations of AdS_3: $k = -1$ corresponds to global coordinates, whereas $k = 0$ and $k = +1$ yield the metrics for massless and massive BTZ black holes [5] (after φ is periodically identified — see next section). This case with $\lambda > 0$ will be the primary focus of our investigations below.

In their constructions, Randall and Sundrum introduced branes on Poincaré invariant slices of AdS, e.g. $r =$ constant in eq. (2.6) with $\lambda = 0$, $k = -1$. There were two distinct scenarios. Originally they considered compactifying the transverse dimension by the introduction of two branes on either end of a finite interval in AdS [6]. In this case, one finds that the inner brane must have a negative tension. However, in a second scenario, they also found lower dimensional gravity arose for a single brane and an infinite transverse dimension [2]. These discussions were extended to de Sitter slicings to give a description of inflationary cosmology within the context of the Randall-Sundrum constructions [12]. In this case, one may again consider two distinct scenarios. Either one has two branes bounding a finite interval, one with positive tension and the other with negative tension, or one has a single brane. In the latter case, however, the transverse direction is not infinite since the brane is spherical and encompasses a finite region at the center of AdS. Following the discussion above, these constructions would correspond to placing the branes along surfaces of constant r with $\lambda < 0$ in eq. (2.6).

Finally one may consider the Randall-Sundrum construction for branes with negative curvature. A subtlety arises in this case which we will illustrate explicitly in four dimensions, but the same comments also apply for five or higher dimensions. In eq. (2.6) with $\lambda > 0$, g_{rr} is singular at $r = \sqrt{\lambda}\ell_4$ but this is simply a coordinate singularity [11]. The latter is most easily seen by making the coordinate transformation $r = \sqrt{\lambda}\ell_4 \cosh u$, with which eq. (2.6) becomes

$$ds^2 = \ell_4^2 du^2 + \lambda\ell_4^2 \cosh^2 u \left[-(\lambda\rho^2 - k)dt^2 + \frac{d\rho^2}{\lambda\rho^2 - k} + \rho^2 d\varphi^2\right]. \tag{2.8}$$

Here we see the entire spacetime is covered by $-\infty < u < \infty$. However there are now two asymptotic regions where the scale factor for the negative curvature slices grows without bound, i.e. $u \to \pm\infty$. Consider introducing a single brane by cutting along a surface of constant u, removing the region with large positive u and gluing an identical copy of the remaining geometry. The resulting space would still have two asymptotic regions as $u \to -\infty$ on either side of the single brane. In this situation, the 'zero mode' bound to this brane would not be normalizable. Alternatively, one might say that it would require an infinite amount of energy to excite this mode [1]. In any event, one would not find an effective lower dimensional theory of gravity on the brane. Hence for the negative curvature slicings, one must introduce two branes by making two cuts and removing both asymptotic regions at $u \to \pm\infty$. Gluing this geometry to an identical copy of itself produces a space where the gravitational 'zero mode' would be normalizable and one would find gravity is effectively lower dimensional on large distance scales.

Another interesting feature arises here because with the negative curvature slicing, the AdS space effectively contains a 'throat'. That is, the scale factor multiplying the metric on each slice, i.e. $r^2 = \lambda\ell_4^2\cosh^2 u$, decreases monotonically to a minimum value at $r = \sqrt{\lambda}\ell_4$ (or $u = 0$) as we approach this surface from either of the asymptotic regions. Hence if we cut the metric (2.8) along one surface at constant positive u and the other at constant negative u, and remove both asymptotic regions at large $|u|$, then the extrinsic curvature on both ends will be a positive multiple of the metric. Hence both of the branes in this construction will have a positive tension, in contrast to the cases with $\lambda \leq 0$, or with $\lambda > 0$ when both cuts are made in the positive u region.[2]

For a Randall-Sundrum construction with two negative curvature branes, we now compute the relationship between the four-dimensional Newton's constant and that for the effective three-dimensional theory of gravity on the branes. The metrics under consideration are of the form

$$ds^2 = \frac{dr^2}{r^2/\ell_4^2 - \lambda} + \frac{r^2}{L_1^2}\, g_{\mu\nu}(x)\, dx^\mu dx^\nu \ . \tag{2.9}$$

We are assuming that one of the branes lies on the slice at $r = L_1$ (on the positive u patch), so $g_{\mu\nu}$ is precisely the induced metric on this brane. This metric will satisfy $\mathcal{R}_{\mu\nu} = -(2\lambda/L_1^2)\, g_{\mu\nu}$. We will assume that the second brane lies at $r = L_2$ on the patch with $u < 0$ (and so the induced metric there is $(L_2/L_1)^2 g_{\mu\nu}$). Now as in standard Kaluza-Klein compactification, we have to integrate over the volume between the two branes. Hence we integrate r from L_1 to the minimum radius $\sqrt{\lambda}\ell_4$ and then back out to L_2 (on the negative u patch). Since our interest is in relating G_3 to G_4, we focus on the Einstein-Hilbert term in the action. The brane tensions

[2]The 'throat' also allows for other exotic constructions. For example, one could produce a single brane by cutting at $u = \pm U_0$ and identifying these two surfaces.

will be tuned in order to produce the required cosmological constant on the brane. The reduced effective action is then

$$\begin{aligned} I' &= \frac{2}{16\pi G_4}\left(\int_{\sqrt{\lambda}\ell_4}^{L_1} + \int_{\sqrt{\lambda}\ell_4}^{L_2}\right) dr \frac{r/L_1}{\sqrt{r^2/\ell_4^2 - \lambda}} \int d^3x \sqrt{-g}\,\mathcal{R}(g) + \cdots \\ &= \frac{1}{16\pi G_3}\int d^3x \sqrt{-g}\left(\mathcal{R}(g) + \frac{2\lambda}{L_1^2}\right). \end{aligned} \tag{2.10}$$

The radial integral is easily evaluated and we obtain

$$2\frac{\ell_4}{L_1}\left(\sqrt{L_1^2 - \lambda\ell_4^2} + \sqrt{L_2^2 - \lambda\ell_4^2}\right) G_3 = G_4\,. \tag{2.11}$$

Note that we have inserted the appropriate cosmological constant term in the three-dimensional effective action, where the three-dimensional AdS scale is $\ell_3 = L_1/\sqrt{\lambda}$. We emphasize that this calculation is normalized for physicists living on the brane at $r = L_1$.

Now we turn to applying the Randall-Sundrum construction to the general solution (2.1) with $m > 0$. It is clear from the $m = 0$ discussion that we have to introduce two different branes. In fact, the asymptotic region far from the black hole corresponds to $x, y \to 0$. It follows from (2.2) and (2.3) that in this limit, the effects of the black hole become negligible and the spacetime approaches (2.6). As usual, one can introduce a two-brane on surfaces where the extrinsic curvature is proportional to the metric. So we need to find two such surfaces in the AdS C-metric (2.1). For a surface of constant x, this will be true when $G'(x) = 0$. The simplest choice (and the one used in [1]) is to take the surface $x = 0$. Similarly, one can use a surface of constant y provided that $H'(y) = 0$. The simplest choice is again $y = 0$. So to construct our solution we take the points (x, y) with $x \geq 0$ and $y \leq 0$ in (2.1) and glue this region to an identical copy along the boundaries $x = 0$ and $y = 0$.[3] It is natural to foliate the quadrant $x \geq 0$, $y \leq 0$ with surfaces of constant slope x/y. Since r is a function only of x/y (2.5), these are surfaces of constant r.[4] The surface $x = 0$ corresponds to $r = 1/A$ and $y = 0$ corresponds to $r = \sqrt{\lambda}/A$. However, note that these surfaces are on opposite sides of the 'throat'. This can be seen in eq. (2.5) where as x/y decreases from zero to minus infinity, r decreases from $1/A$ to its minimum value, and then increases back to $\sqrt{\lambda}/A$. So these surfaces both have positive extrinsic curvature (this can also be checked directly) and both correspond to positive tension branes. Notice that the brane at $x = 0$ cuts through the black hole horizon, and contains (for $m \neq 0$) the singularity at $y = -\infty$. In contrast, the brane at $y = 0$ is completely non-singular and does not contain any black hole

[3] Other possible locations for the branes will be considered briefly in the discussion section.

[4] Even though r was originally defined for $m = 0$, it is convenient to introduce it even for $m > 0$. This is because the solution far from the black hole always approaches the AdS_4 metric (2.6).

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horizons. These choices for the two branes correspond to $L_1 = 1/A$ and $L_2 = \sqrt{\lambda}/A$ in the previous paragraph. Using eq. (2.4), a bit of algebra then reduces eq. (2.11) to:

$$G_3 = \frac{AG_4}{2}\,. \tag{2.12}$$

Notice that this is independent of λ and curiously, it agrees with the relation derived with a single brane and $\lambda = 0$, i.e. $A = 1/\ell_4$ [1]. Further with the above choice of L_1, the effective three-dimensional AdS scale is given by

$$\ell_3^2 = \frac{1}{A^2\lambda}\,. \tag{2.13}$$

Note that $\ell_3 > \ell_4$, where the four-dimensional AdS scale is given in eq. (2.4). It will be useful to keep in mind that the parameters A and λ together determine the cosmological constant on the brane and the bulk, while m is related to the (localized) black hole on the brane, which we discuss below.

3. Black holes on the brane

The AdS C-metric with $\lambda = 0$ was discussed extensively in [1]. Here we examine the case $\lambda > 0$.

3.1 BTZ black strings

As noted above, when $m = 0$ the metric reduces to eq. (2.6) and if $k = +1$, surfaces of constant r have the geometry of BTZ black holes [5] when we simply make a periodic identification of φ with some period $\Delta\varphi$. The horizon is at $\rho = 1/\sqrt{\lambda}$. The entire spacetime describes a 'BTZ black string' in AdS_4. When the branes are introduced as above, the black string extends between $r = L_1 = 1/A$ on the positive u patch and $r = L_2 = \sqrt{\lambda}/A$ on the negative u patch. These black strings are analogous to those constructed in [7], except that we are considering a construction with two branes.

Including the contributions from the two AdS geometries that have been glued together, the total area of the event horizon is

$$\mathcal{A} = \frac{2\Delta\varphi}{A^2\sqrt{\lambda}}\,. \tag{3.1}$$

Note that at $r = L_1$, the horizon has a proper circumference of $\mathcal{C}_1 = \Delta\varphi/A\sqrt{\lambda}$. The circumference shrinks to $\mathcal{C}_{\rm min} = \Delta\varphi/A\sqrt{\lambda+1}$ at $r = r_{\rm min} = \sqrt{\lambda}\ell_4$, and then expands again to $\mathcal{C}_2 = \Delta\varphi/A$ as the string extends to $r = L_2$.

To determine the effective mass, we compare the induced metric at $r = L_1$ with the standard BTZ metric [5][5]

$$ds^2 = -\left(\frac{\hat\rho^2}{\ell_3^2} - 8G_3M_3\right)d\hat t^2 + \left(\frac{\hat\rho^2}{\ell_3^2} - 8G_3M_3\right)^{-1}d\hat\rho^2 + \hat\rho^2 d\hat\varphi^2 \tag{3.2}$$

[5]In the literature on BTZ black holes, it is common to set $G_3 = 1/8$.

where $\hat{\varphi}$ has periodicity 2π, and ℓ_3 is the three-dimensional radius of curvature (2.13). To produce this form, we must rescale the coordinates in eq. (2.6) as

$$t = \frac{A\Delta\varphi}{2\pi}\hat{t}\,, \qquad \rho = \frac{2\pi A}{\Delta\varphi}\hat{\rho}\,, \qquad \varphi = \frac{\Delta\varphi}{2\pi}\hat{\varphi}\,. \tag{3.3}$$

A short calculation then shows that the total three-dimensional mass is

$$M_3 = \frac{1}{8G_3}\left(\frac{\Delta\varphi}{2\pi}\right)^2. \tag{3.4}$$

Note that the metric (3.2) has a conical-like singularity at $\hat{\rho} = 0$. This singularity then appears on each constant r slice of the BTZ black string and so extends throughout the four-dimensional spacetime.

3.2 BTZ-like black holes

We are, however, more interested in solutions with localized four-dimensional black holes on the brane, so we turn to the case with $m > 0$. The case $m = 0$ and $k = 1$ considered above was special in that $G(x)$ never vanished, so φ could have any periodicity. For $m > 0$, this periodicity is fixed by demanding that the geometry is smooth on the axis where $G(x)$ vanishes. With $m > 0$, $G(x)$ always has one (and only one) positive root, x_2, and a conical singularity on this axis is avoided by setting $\Delta\varphi$ to

$$\Delta\varphi = \frac{4\pi}{|G'(x_2)|}\,. \tag{3.5}$$

Recall that in our construction of a black hole localized on a brane, the portion of the space with $x < 0$ is cut out and so x is restricted to lie in the range $0 \le x \le x_2$. Now if we let $\rho = -1/y$ the geometry induced on the two-brane at $x = 0$ is

$$ds^2 = \frac{1}{A^2}\left[-\left(\lambda\rho^2 - k - \frac{2mA}{\rho}\right)dt^2 + \left(\lambda\rho^2 - k - \frac{2mA}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\varphi^2\right]. \tag{3.6}$$

This is similar to the locally AdS metrics, but with extra terms of the form $2mA/\rho$ coming from the four-dimensional nature of the black hole. While these corrections have power law decay in ρ, they are actually decaying exponentially in the radial proper distance which is asymptotically $R_{\text{proper}} = \ell_3 \ln \rho$. This should be expected since we have compactified the transverse direction by introducing two branes. Hence, we should find a discrete spectrum of massive Kaluza-Klein modes for the fluctuations of the four-dimensional metric [6]. This behavior can be contrasted to that for asymptotically flat branes with an infinite transverse dimension. There one finds power law corrections to the metric [1], or to the newtonian potential [2, 13, 14], arising from a continuum of massive 'Kaluza-Klein' modes.

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As $\rho \to \infty$ we recover the geometry of the surface $r = 1/A$ in (2.6). By rescaling the coordinates as in eq. (3.3), we can compare this asymptotic geometry to the standard BTZ form (3.2) in order to determine the three-dimensional mass. The result of this calculation is

$$M_3 = \frac{k}{8G_3}\left(\frac{\Delta\varphi}{2\pi}\right)^2 . \tag{3.7}$$

Using (3.5) and (2.3), it is useful to re-express this mass as

$$M_3 = \frac{k}{2G_3}\frac{x_2^2}{(3 + k\,x_2^2)^2} . \tag{3.8}$$

Now the horizon of the four-dimensional black hole is at the negative root of $H(y)$, $y = y_0$. It extends off the brane at $x = 0$ to the maximum value of x, x_2. Using (2.5), the latter corresponds to the radial coordinate

$$r_0 = \frac{\sqrt{y_0^2 + \lambda x_2^2}}{A(x_2 - y_0)} . \tag{3.9}$$

The area of the horizon is finite and equal to

$$\mathcal{A} = \frac{2\Delta\varphi}{A^2}\int_0^{x_2}\frac{dx}{(x-y_0)^2} = \frac{2\Delta\varphi}{A^2}\frac{x_2}{|y_0|(x_2 + |y_0|)} . \tag{3.10}$$

Finally note that the brane metric (3.6) contains a curvature singularity at $\rho = 0$. This singularity, however, is confined to the brane, and is not present on test two-branes at $r \neq 1/A$, i.e. in the induced metric on constant r slices for $r \neq 1/A$.

Now recall that k can take the values 0 or ± 1. So we discuss each of these individual cases in turn below:

(i) The case $k = +1$. This is the most interesting case since with $k = 1$, eq. (3.6) is similar to the BTZ metric, but with extra mA/ρ corrections. If $2mA \ll \lambda^{-1/2}$, these extra terms are negligible outside the horizon, and so the exterior geometry is essentially identical to that of a BTZ black hole. However, for $2mA \gg \lambda^{-1/2}$, there will be significant deviations from the BTZ metric outside the black hole.

Surprisingly, we see from eq. (3.8) that M_3 vanishes both when $mA \to 0$ ($x_2 \to \infty$) and when $mA \to \infty$ ($x_2 \to 0$). This implies that, in contrast to BTZ strings, there is a maximum mass for the black holes that are localized to the brane. The maximum mass occurs at $mA = 2/\sqrt{27}$ ($x_2 = \sqrt{3}$), where

$$M_{3,\mathrm{max}} = \frac{1}{24G_3} . \tag{3.11}$$

This is a very small mass of the order the three-dimensional Planck mass. The effect is entirely due to four-dimensional dynamics. It would appear that one cannot localize a large mass black hole on the brane. The only solution for large mass would the BTZ black string. Even though M_3 vanishes when $mA \to 0$, it is not proportional to m even in this limit. For small mA, $x_2 \approx 1/(2mA)$, and so we have

$$G_3 M_3 = 2(mA)^2. \tag{3.12}$$

There are two limits in which we can easily examine the geometry of the black hole horizon: small mA, $2mA \ll \min\{1, \lambda^{-1/2}\}$, and large mA, $2mA \gg \max\{1, \lambda^{-1/2}\}$.

For small mA, $x_2 \approx 1/2mA$ and $y_0 \approx -\sqrt{\lambda}$. In this limit, the area of the event horizon (3.10) approaches that of the black string with the same mass (same $\Delta\varphi$). This is less surprising if one observes that as mA goes to zero, x_2 goes to infinity, and r_0 approaches the position of the second brane at $r = \sqrt{\lambda}/A$. That is, the horizon nearly stretches across the bulk AdS space to the second brane. With a closer examination, one realizes that outside the horizon, $y_0 \le y < 0$, the function $H(y)$ is essentially unchanged by a small mA. Furthermore, $G(x)$ is unchanged until x becomes of order $1/mA$, at which point it rapidly goes to zero. So the geometry near these 'localized' black holes is very similar to that of the BTZ black string off the brane. In fact, a small 'localized' black hole is really a black string which is capped off at the ends![6] One way of understanding the fact that small BTZ-like black holes cannot be localized near the brane very well is just that there are no small black holes in AdS_4 which look like BTZ on a slice (see the case $k = -1$ below).

For large mA, $x_2 \approx (2mA)^{-1/3}$ and $y_0 \approx -(\lambda/2mA)^{1/3}$. Above, it was noted that the mass measured on the brane goes to zero in this limit. However, the black hole area tends to a finite value

$$\mathcal{A} = \frac{8\pi}{3A^2} \frac{1}{\lambda^{1/3}(1 + \lambda^{1/3})}. \tag{3.13}$$

Similarly, the circumference of the horizon on the brane, $\mathcal{C} \to 4\pi/3A\lambda^{1/3}$. Further in this limit, $r_0 \approx \lambda^{1/3}/A\sqrt{1+\lambda^{1/3}}$ and the proper distance that the black hole extends off the brane is finite. Note that all of these quantities depend only on λ, but not on m. The interpretation of the result for r_0 depends on the value of λ. First note that in general $r_{\min} = \sqrt{\lambda}\ell_4$ is achieved in eq. (2.5) at $x = |y|/\lambda$ with negative y. Hence given the values of x_2 and y_0 above, we see that for small λ, r_0 actually sits on the positive u side of the 'throat' and so the event horizon has a pancake geometry. As λ increases, the black hole becomes fatter and touches the 'throat' at $\lambda = 1$. For large λ, the horizon extends through the 'throat' onto the negative u patch.

Of course, the most surprising effect found in the large mA regime is that one can find black holes of finite size with vanishingly small mass on the brane. This is reminiscent of the fact that for asymptotically flat branes, $\lambda = 0$, one can construct arbitrarily large black holes with finite energy [1]. This seemed to be related to the fact that there is a maximum total mass in $2+1$ gravity without a cosmological

[6] These black holes would then realize the 'cigar-shaped' horizon geometries envisioned in ref. [7].

constant. We now see that even though there is no a priori maximum mass when the cosmological constant is negative, there is still an upper limit on the mass of a localized black hole on the brane. This results in finite black holes with arbitrarily small mass.

In the section 4.1, we will examine the stability of these BTZ-like black holes as compared to the BTZ strings, on the basis of their relative entropies.

(ii) The case $k = -1$. With $k = -1$, the metric on the brane (3.6) is then identical to a slice through the equator of the (spherically symmetric) Schwarzschild AdS metric. The situation is now similar to the case of an asymptotically flat brane [1]: When mA is small, the horizon is approximately spherical, but when mA is large, the black hole is a flattened pancake. Hence in these geometries, the black hole is very effectively confined to the vicinity of the brane. However, one crucial difference is that the black holes with $k = -1$ have *negative* mass on the brane, as seen in eq. (3.7). Note that the mass of these black holes varies monotonically from $M_3 = 0$ for $x_2 = 0$ ($mA \to \infty$) to $M_3 = -1/8G_3$ for $x_2 = 1$ ($mA \to 0$). Precisely this range of masses arise in the negative mass solutions in AdS_3 constructed in [5]. This sector interpolates between the global ground state of AdS_3 (with mass $M_3 = -1/8G_3$) and the massless BTZ black hole. In pure $2+1$ gravity these solutions have (naked) conical singularities. In the present context, these singularities can be hidden behind the horizons of four-dimensional black holes.

(iii) The case $k = 0$. The $k = 0$ solutions are very similar to the case of large mA and $k = +1$ (or $k = -1$). This is not new: it is known that in AdS space, the horizons of static spherical or hyperbolic black holes become flat in the limit of infinite mass (this is sometimes called the "infinite volume limit", particularly in the context of the AdS/CFT duality). In the present situation, this can be achieved by scaling the mass and the coordinates as $(x, y, t) \to \alpha^{-1/3}(x, y, t)$, $m \to \alpha m$, and letting $\alpha \to \infty$.

4. Black hole thermodynamics

We would now like to consider the thermodynamic properties of these BTZ-like black holes on AdS_3 branes. Consider first the Hawking temperature. As usual we can define the Hawking temperature as $T = \kappa/2\pi$ where κ is the surface gravity of the black hole. There is a small difficulty since κ depends on the normalization of the timelike Killing vector $\xi \propto \partial_t$. In asymptotically flat spacetimes, one usually requires that ξ have unit norm at infinity, but it is not obvious what the proper normalization is in the present case where the branes are asymptotically AdS. To resolve this difficulty, we can turn to the thermodynamics of the BTZ black hole (3.2). The circumference of the horizon is $\mathcal{C} = 2\pi r_H = 4\pi\ell_3\sqrt{2G_3M_3}$. Hence the entropy satisfies

$$S = \pi \ell_3 \sqrt{\frac{2M_3}{G_3}} \quad \rightarrow \quad \frac{dS}{dM_3} = \frac{\pi \ell_3}{\sqrt{2G_3 M_3}} = \frac{1}{T} . \tag{4.1}$$

Now one finds that this temperature matches precisely the surface gravity of the Killing vector $\xi = \partial_t$, or alternatively $\beta = 1/T$ is the periodicity of the euclidean coordinate $\tau = it$. Hence this suggests that if we normalize the coordinates such that the asymptotic metric on the AdS_3 brane is

$$ds^2 \simeq -\frac{\hat{\rho}^2}{\ell_3^2} d\hat{t}^2 + \frac{\ell_3^2}{\hat{\rho}^2} d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\varphi}^2 , \tag{4.2}$$

where $\hat{\varphi}$ has periodicity 2π, then the Hawking temperature will be the surface gravity associated with the Killing vector $\partial_{\hat{t}}$ in this coordinate system. So having decided on the appropriate normalization, a short calculation shows that

$$T = \frac{A}{2\pi} \frac{|H'(y_0)|}{|G'(x_2)|} = \frac{A}{2\pi} \frac{|y_0|}{x_2} \frac{3mA|y_0| + 1}{3mAx_2 - 1} . \tag{4.3}$$

The black hole entropy is simply determined by the four-dimensional area (3.10)

$$S = \frac{\mathcal{A}}{4G_4} = \frac{\Delta\varphi}{2G_4 A^2} \frac{x_2}{|y_0|(x_2 + |y_0|)} . \tag{4.4}$$

4.1 Equality of three and four dimensional masses

For the case of asymptotically flat branes, it was shown [1] that the three-dimensional mass measured asymptotically on the brane agreed precisely with a four-dimensional thermodynamic mass obtained by integrating the first law

$$\delta M = T\, \delta S . \tag{4.5}$$

We now show that this is also true when there is a negative cosmological constant on the brane. It is impractical to try to express T in terms of S in order to compute M. Instead, it proves more convenient to express both S and T in terms of a single auxiliary variable. An appropriate choice is

$$\hat{z} = \frac{|y_0|}{x_2} , \tag{4.6}$$

for which one has various identities such as

$$\begin{aligned} x_2 &= \frac{1}{2mA} \frac{\lambda + \hat{z}^2}{\lambda - \hat{z}^3} , \\ |y_0| &= \frac{\hat{z}}{2mA} \frac{\lambda + \hat{z}^2}{\lambda - \hat{z}^3} , \\ 2mA &= \frac{(\lambda + \hat{z}^2)\hat{z}\sqrt{1 + \hat{z}}}{(\lambda - \hat{z}^3)^{3/2}} . \end{aligned} \tag{4.7}$$

Note that $\hat{z}$ is a monotonic function of mA, and that

$$\begin{aligned} mA &\to 0\,, \qquad && \hat{z} \simeq 2mA\sqrt{\lambda} \\ mA &\to \infty\,, \qquad && \hat{z} \simeq \lambda^{1/3}\,. \end{aligned} \tag{4.8}$$

One also has that x_2 and $|y_0|$ are monotonic functions of $\hat{z}$ in the range $0 \le \hat{z} \le \lambda^{1/3}$.

Now in terms of $\hat{z}$, the Hawking temperature (4.3) becomes

$$T = \frac{A\hat{z}}{2\pi}\frac{2\lambda + 3\lambda\hat{z} + \hat{z}^3}{\lambda + 3\hat{z}^2 + 2\hat{z}^3}\,. \tag{4.9}$$

Let us also consider

$$\frac{\partial T}{\partial \hat{z}} = \frac{A}{\pi}\frac{(\lambda - \hat{z}^3)(\lambda + 3\lambda\hat{z} - 3\hat{z}^2 - \hat{z}^3)}{(\lambda + 3\hat{z}^2 + 2\hat{z}^3)^2}\,. \tag{4.10}$$

From the latter we see that $\partial T/\partial\hat{z} = 0$ at $\hat{z} = \lambda^{1/3}$. For $\lambda < 1$, $\partial T/\partial\hat{z}$ also has another zero in the physical range $0 \le \hat{z} \le \lambda^{1/3}$. Hence for $\lambda \ge 1$, T is a monotonic function increasing from 0 at $\hat{z} = 0$ to

$$T\left(\hat{z} = \lambda^{1/3}\right) = \frac{A}{2\pi}\lambda^{2/3}\,. \tag{4.11}$$

For $\lambda < 1$, T rises from 0 at $\hat{z} = 0$ to a maximum at some intermediate value of $\hat{z}$, and then decreases down to a local minimum at $\hat{z} = \lambda^{1/3}$ given by the same formula as in eq. (4.11).

In terms of $\hat{z}$, the entropy (4.4) becomes

$$S = \frac{2\pi}{G_4 A^2}\frac{\hat{z}}{\lambda + 3\hat{z}^2 + 2\hat{z}^3}\,. \tag{4.12}$$

Here again we also consider

$$\frac{\partial S}{\partial \hat{z}} = \frac{2\pi}{G_4 A^2}\frac{\lambda - 3\hat{z}^2 - 4\hat{z}^3}{(\lambda + 3\hat{z}^2 + 2\hat{z}^3)^2}\,. \tag{4.13}$$

The latter always has a zero in the range $0 \le \hat{z} \le \lambda^{1/3}$. Hence the entropy increases from zero at $\hat{z} = 0$ to a maximum at some intermediate value of $\hat{z}$, and then decreases down to

$$S = \frac{2\pi}{3G_4 A^2}\frac{1}{\lambda^{1/3}(1 + \lambda^{1/3})} \tag{4.14}$$

at $\hat{z} = \lambda^{1/3}$, in agreement with (3.13).

Using the first law (4.5), we can now integrate

$$\frac{\partial M_4}{\partial \hat{z}} = T\frac{\partial S}{\partial \hat{z}}\,, \tag{4.15}$$

with the boundary condition that $M_4 = 0$ at $\hat{z} = 0$. A short calculation yields

$$M_4 = \frac{1}{G_4 A}\frac{\hat{z}^2(1 + \hat{z})(\lambda - \hat{z}^3)}{(\lambda + 3\hat{z}^2 + 2\hat{z}^3)^2}\,. \tag{4.16}$$

We want to compare this now to (3.8). Expressing the latter in terms of $\hat{z}$, one finds

$$M_3 = \frac{1}{2G_3} \frac{\hat{z}^2(1+\hat{z})(\lambda - \hat{z}^3)}{(\lambda + 3\hat{z}^2 + 2\hat{z}^3)^2} \tag{4.17}$$

so that using (2.12) we get precise agreement $M_3 = M_4$, just as was found in [1] for flat branes.

The thermodynamical analysis for $k = -1$ can be carried through in close analogy to the analysis for BTZ black holes above. As a matter of fact, if we express the mass, entropy and temperature in terms of the variable $\hat{z}$ of (4.6) we obtain precisely the same results (4.9), (4.12), (4.16), and (4.17), only this time we have $\hat{z}^3 > \lambda$. In fact, we can also recover the results for $k = 0$ by setting $\hat{z}^3 = \lambda$. Hence, letting $\hat{z}$ vary over $0 \le \hat{z} < \infty$, these formulas give us M, S and T for the full range of black holes with $\lambda > 0$ and arbitrary k.

If we compute the thermodynamic mass M_4 of the BTZ black strings discussed above, we again find $M_3 = M_4$. To see this, notice that for a black string, the four-dimensional entropy computed from the horizon area (3.1) agrees with the three-dimensional entropy computed from the circumference (4.1):

$$S = \frac{\mathcal{A}}{4G_4} = \frac{\pi}{A}\sqrt{\frac{2M_3}{\lambda G_3}} = \frac{\mathcal{C}}{4G_3}\,, \tag{4.18}$$

where we have used (2.12) and the fact that the geometry on the brane is exactly (3.2) with $\ell_3 = 1/A\sqrt{\lambda}$. The temperature on the brane must also agree with the temperature in the bulk since it is constant over the horizon. Since the temperature and entropy both agree, the mass obtained by integrating the first law must also agree.

4.2 Stability analysis

In the range $0 \le M_3 \le 1/24G_3$, there are two localized black hole solutions as well as the black string. Which of these solutions gives the correct stable configuration for a given mass? In [7] it was suggested that the Gregory-Laflamme instability [15] should play an important role in determining the stable solution. Gregory and Laflamme observed, in the context of Kaluza-Klein compactification, that when a black string becomes longer than its transverse size, unstable modes arise which appear to lead to the black string breaking up into localized black holes. In [7], this result was combined with the observation that in AdS space the only metric fluctuations that can be supported have proper wavelengths shorter than the AdS scale ℓ. Hence it was argued that when the transverse size of a black string reaches ℓ, the Gregory-Laflamme instability should set in and cause the black string to break up forming a localized black hole. Their arguments were made in the context of discussing black strings and black holes in the Randall-Sundrum scenario [2] with an infinite transverse dimension, but presumably they extend to the other scenario [6] with a finite transverse dimension. This is the context in which we would like to consider this mechanism here. Because of the negative curvature on the branes, the minimum transverse size of the black string actually arises at the 'throat', i.e. $r = \sqrt{\lambda}\ell_4$, rather than at the second brane. In any event, the fact that the black strings only shrink to a minimum size (which is some fixed fraction of the transverse size on the brane) provides an explanation as to why black strings should appear as the only (stable) solution for large masses. This result is in fact very similar to ordinary Kaluza-Klein compactification where black strings prove to be the appropriate solution for large masses. Essentially there, localized black holes with large masses do not fit inside the compact dimensions.

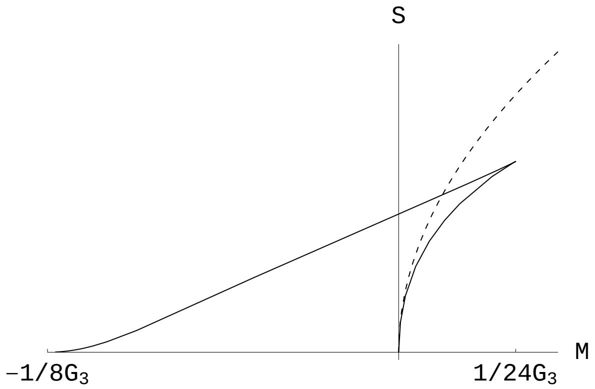

Figure 1: Entropy versus mass for BTZ black strings (dashed line), BTZ-like black holes (solid line, $M \geq 0$), and $k = -1$ localized black holes in AdS_3 (solid line, $M < 0$). The black hole curve starts at $S = M = 0$ for $mA = 0$, and then grows to a maximum of the mass and entropy for $mA = 2/\sqrt{27}$. As $mA \to \infty$ the solutions tend to $M = 0$, S finite (eq. (4.14)).

However, presumably the range $0 \leq M_3 \leq 1/24G_3$ is a regime[7] where the black strings may break up and become localized. Rather than attempting a detailed fluctuation analysis [15], we will argue stability of a given solution on the grounds that it maximizes the entropy for a given mass. For small M_3, the black string entropy (4.18) is approximately equal to that of black holes with small mA. In accord with the physical picture of the horizon of these black holes found in section 3.2, one can show that the black hole entropy is slightly smaller than that of the black string. From figure 1 one sees that in fact the black string entropy exceeds that for all of the black holes in the range $0 \leq mA \leq 2/\sqrt{27}$ — recall that the upper limit corresponds to the maximum mass (3.11). On the other hand, for large mA, M_3 approaches zero again while the entropy has the finite limit in eq. (4.14), in contrast to the vanishing entropy of the black strings. Hence for small M_3, these localized black holes will provide the stable configuration. This is presumably a situation where the strings are unstable and pinch-off to form a black hole. In order to verify the picture put forward in ref. [7], one should determine the minimum mass where the black string entropy still dominates over that of the black holes. One could then verify that this crossover occurs when the minimum transverse size of the black strings reaches the AdS scale.

[7]Recall this regime corresponds to the $k = +1$ black holes.

At the crossover point, the black string and black hole masses, eqs. (3.4) and (3.7) (with $k = +1$), are identified and hence $\Delta\varphi$ is the same for both solutions. Also equating the black hole and black string entropies, eqs. (4.12) and (4.18) (substituting eq. (4.17) in the latter), yields a fourth order polynomial in terms of the auxiliary variable

$$\hat{z}(\lambda - \hat{z}^2 - \hat{z}^3) = 0\,. \tag{4.19}$$

Here, the root $\hat{z} = 0$ simply indicates that the mass and entropy of the small black holes matches that of the black strings. Denoting the solution of the crossover point as $\hat{z}_x$, it satisfies

$$\hat{z}_x^2 + \hat{z}_x^3 = \lambda\,. \tag{4.20}$$

While it would be possible to analytically determine the solution of this equation, we did not find such an explicit solution very illuminating. The circumference of the black string at the 'throat' was calculated to be $C_{\rm min} = \Delta\varphi/A\sqrt{\lambda+1}$ in section 3.1. Hence using eq. (2.4), we have

$$\frac{C_{\rm min}}{2\pi\ell_4} = \frac{\Delta\varphi}{2\pi} = \frac{2\sqrt{\lambda}\hat{z}_x}{3\lambda + \hat{z}_x^2}\,. \tag{4.21}$$

Using eq. (4.20), this result is implicitly a function of only λ. Let us consider the result for small λ for which $\hat{z}_x^2 \approx \lambda$ and hence eq. (4.21) yields

$$\frac{C_{\rm min}}{2\pi\ell_4} \approx \frac{1}{2} \qquad \text{for small } \lambda\,. \tag{4.22}$$

So at the transition point, the minimum size of the black string horizon is indeed of order the AdS_4 scale. This result is certainly consistent with the instability mechanism suggested in ref. [7]. On the other hand, for large λ, one has $\hat{z}_x^3 \approx \lambda$ and hence

$$\frac{C_{\rm min}}{2\pi\ell_4} \approx \frac{2}{3\lambda^{1/6}} \ll 1 \qquad \text{for large } \lambda. \tag{4.23}$$

Hence we seem to have found an inconsistency with the discussion of ref. [7].

What has gone wrong? To resolve this apparent contradiction, we take a closer look at the spacetime geometry. First recall

$$\ell_4 = \frac{1}{A\sqrt{\lambda+1}}\,, \qquad L_1 = \frac{1}{A}\,, \qquad L_2 = \frac{\sqrt{\lambda}}{A}\,. \tag{4.24}$$

Now in the regime of small λ, we have

$$L_1 \approx \ell_4 \gg L_2\,, \tag{4.25}$$

and hence the first brane at $r = L_1$ plays the role of the Planck brane, i.e. it inherits the larger scale factor from the AdS geometry. Thus our construction and analysis are consistent with the suggested instability mechanism [7]. That is, as the mass of the black string shrinks (by, e.g., Hawking radiation) eventually its minimum cross-section reaches the AdS scale ℓ_4. At this point, the Gregory-Laflamme instability causes the string to break up and form a black hole localized in the vicinity of the Planck brane at $r = L_1$. On the other hand, in the regime of large λ, we have

$$L_2 \gg L_1 \gg \ell_4\,, \tag{4.26}$$

and hence the second brane at $r = L_2$ plays the role of the Planck brane. Hence using the mechanism of [7], we would argue that as the mass of the black string shrinks, it would become unstable when $C_{\rm min}/2\pi\ell_4 \approx 1$ and the string would then break up to form a black hole localized at the second brane, *not* the brane at $r = L_1$. Hence while the result in eq. (4.23) is not incorrect, it is probably not relevant to determining the stable configuration in this situation. That is we are comparing the black string to the 'wrong' black hole solution. Instead we expect the true stable solution to be a black hole localized on the brane at $r = L_2$, but our construction in section 3 has not provided us with this solution.

In short, in the regieme where we can reliably compute the transition between the black string and black hole, our results are consistent with the estimates in [7].

5. Adding rotation

It is easy to include the effects of rotation into the picture. What we need is a metric that describes an accelerating, rotating black hole in a spacetime with a negative cosmological constant. Such a solution can be found as a member of a subfamily of solutions of the most general type D metric[8] constructed in [8] (the complete family of neutral solutions possesses, in addition, a NUT parameter, which we have set to zero here). The solution we are interested in is

$$\begin{aligned} ds^2 = \frac{1}{A^2(x-y)^2}\Big[&\frac{H(y)}{1+a^2x^2y^2}(dt+ax^2d\varphi)^2 - \frac{1+a^2x^2y^2}{H(y)}dy^2 + \\ &+ \frac{1+a^2x^2y^2}{G(x)}dx^2 + \frac{G(x)}{1+a^2x^2y^2}(d\varphi - ay^2dt)^2\Big]\,, \end{aligned} \tag{5.1}$$

where

$$H(y\ \ ) = -\lambda + ky^2 - 2mAy^3 - a^2y^4\,, \tag{5.2}$$
$$G(x) = 1 + kx^2 - 2mAx^3 + a^2\lambda x^4 \tag{5.3}$$

and $k = +1, 0, -1$. Again, the metric satisfies $R_{AB} = -(3/\ell_4^2)g_{AB}$, with ℓ_4 defined as in (2.4). The AdS C-metric (2.1) is recovered by simply setting $a = 0$.

[8]A type D metric is one in which the Weyl tensor is algebraically special and has two pairs of repeated principal null directions [16]. The "D" stands for "degenerate" and has nothing to do with D-branes or D-terms.

When $m = 0$ the geometry is once again that of AdS_4 in disguise. In this case, the change of coordinates that makes this manifest is

$$r = \frac{\sqrt{y^2 + \lambda x^2}}{A(x-y)}\,, \qquad \rho = \sqrt{\frac{1 + kx^2 - a^2x^2y^2}{y^2 + \lambda x^2}}\,. \tag{5.4}$$

Then (5.1) becomes

$$ds^2 = \frac{dr^2}{r^2/\ell_4^2 - \lambda} + r^2 \left[-\left(\lambda\rho^2 - k + \frac{a^2}{\rho^2}\right) dt^2 + \frac{d\rho^2}{\lambda\rho^2 - k + a^2/\rho^2} + \rho^2 \left(d\varphi - \frac{a}{\rho^2}dt\right)^2 \right]. \tag{5.5}$$

The three-dimensional metrics in brackets have constant Riemann curvature. Notice that, as before, these sections at constant r are sections at constant x/y. Now, however, the coordinate systems are rotating. For $\lambda > 0$ and $k = +1, 0$, these sections are precisely the geometries of rotating BTZ black holes. So the entire spacetime can be viewed as a rotating BTZ black string.

The metrics (5.1) present features familiar from the Kerr metric. For example, when $m \neq 0$ they have a curvature singularity at

$$\frac{1}{y^2} + a^2x^2 = 0\,, \tag{5.6}$$

which is closely analogous to the ring singularity of the Kerr solution at $r^2 + a^2\cos^2\theta = 0$ in standard Boyer-Lindquist coordinates. On the other hand, at the zeroes x_i of $G(x)$ the Killing vector

$$\frac{\partial}{\partial\varphi} - ax_i^2\frac{\partial}{\partial t} \tag{5.7}$$

has a fixed-point set. These are the rotation axes. In order to avoid a conical defect at one of them, say at $x = x_2$, then we have to identify points along the integral curves of (5.7) with the appropriate period. This corresponds to changing coordinates from t, φ to $\tilde{t}, \varphi$ where

$$\tilde{t} = t + ax_2^2\varphi \tag{5.8}$$

and choosing the period of φ to be given by (3.5), on surfaces of fixed $\tilde{t}$.[9]

In a similar way, at the roots y_i of $H(y)$ the Killing vector

$$\frac{\partial}{\partial t} + ay_i^2\frac{\partial}{\partial\varphi} \tag{5.9}$$

becomes null. These correspond to horizons with angular velocity $\Omega = ay_i^2$. Actually, this is appropriate only if we are considering black strings with no rotation axis. As

[9] Of course, periodic identification of φ at constant $\tilde{t}$ is not the same as periodic identification of φ at constant t.

we have just seen, if there is an axis, regularity requires that we use $\tilde{t}, \varphi$ coordinates. In this case, the angular velocity is

$$\tilde{\Omega} = \frac{a y_i^2}{1 + a^2 x_2^2 y_i^2} \,. \tag{5.10}$$

Observe that when $a \neq 0$, $H(y)$ will typically have one more root than in the case $a = 0$. This corresponds to the addition of an inner horizon which is characteristic of rotating black holes.

The construction of a two-brane can be performed, as before, at $x = 0$ (and also at $y = 0$, which will be needed when $\lambda > 0$). To analyze the metric on the brane, let $\rho = -1/y$, so the section at $x = 0$ is

$$ds^2 = \frac{1}{A^2}\Bigg[-\left(\lambda\rho^2 - k - \frac{2mA}{\rho} + \frac{a^2}{\rho^2}\right) dt^2 + \\ + \frac{d\rho^2}{\lambda\rho^2 - k - 2mA/\rho + a^2/\rho^2} + \rho^2\left(d\varphi - \frac{a}{\rho^2}dt\right)^2\Bigg] \,. \tag{5.11}$$

When $\lambda > 0$ and $k = 1$, this is similar to the rotating BTZ metric, only with additional $2mA/\rho$ terms due, as we discussed earlier, to the four-dimensional origin of the black hole. The discussion following (3.6) applies to this situation as well. In the following we focus on the cases of asymptotically flat branes, $\lambda = 0$, and asymptotically AdS_3 branes, $\lambda > 0$.

5.1 $\lambda = 0$: Kerr black holes on the brane

Black holes on asymptotically flat branes were studied, in the absence of rotation, in [1] — the notation used there corresponds to $\mu = mA$ and $\ell = 1/A$. Here we will focus only on the new features introduced by a non-vanishing a.

When $\lambda = 0$ ($A = 1/\ell_4$) we have to set $k = -1$, in order that ∂_t is timelike at infinity on the brane. When $m = 0$, the roots of $G(x)$ are just $x_i = \pm 1$, and $|G'(x_i)| = 2$ so to avoid conical singularities on these axes we need to set $\tilde{t} = t + a\varphi$ as in eq. (5.8) and identify φ with period 2π (3.5). The geometry is that of AdS_4, but instead of the coordinates (5.4) it is more convenient to define

$$z = 1 - \frac{x}{y} \,, \qquad \tilde{\rho} = -\frac{\sqrt{(1-x^2)(1+a^2y^2)}}{y} \,, \tag{5.12}$$

which brings the metric (5.1) into Poincaré coordinates

$$ds^2 = \frac{\ell_4^2}{z^2}(-d\tilde{t}^2 + d\tilde{\rho}^2 + \tilde{\rho}^2 d\varphi^2 + dz^2). \tag{5.13}$$

Apart from z being essentially the inverse of r, the choice of radial coordinate on the brane is slightly different from (5.4) with $\lambda = 0$.

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Now consider the case $m > 0$. As was the case in the absence of rotation, when $\lambda = 0$ the surface $y = 0$ is a degenerate cosmological horizon analogous to $z = \infty$ in (5.13). When $|a| < mA$ there are two other zeroes of $H(y)$, at $y = y_{0\pm} = -mA/a^2 \pm \sqrt{m^2A^2 - a^2}/a^2$, which are interpreted as inner and outer black hole horizons. In the extremal limit $|a| = mA$, they coincide at $y = -1/mA$. If $|a| > mA$ the singularity would be naked.

The metric on the brane at $x = 0$ is locally equivalent to the equatorial slice of the four-dimensional Kerr metric. To see this, set $\hat{t} = (t + a\varphi)/A$, $\hat{\rho} = \rho/A$. Then since $\lambda = 0$ and $k = -1$, (5.11) becomes

$$ds^2 = -\left(1 - \frac{2m}{\hat{\rho}}\right) d\hat{t}^2 + \left(1 - \frac{2m}{\hat{\rho}} + \frac{a^2}{A^2\hat{\rho}^2}\right)^{-1} d\hat{\rho}^2 + \\ + \left(\hat{\rho}^2 + \frac{a^2}{A^2} + \frac{2ma^2}{A^2\hat{\rho}}\right) d\varphi^2 - \frac{4am}{A\hat{\rho}} d\hat{t} d\varphi \,. \tag{5.14}$$

This is indeed the equatorial section of the Kerr solution, with mass $G_4M_4 = m$ and rotation parameter a/A.

Even though the metric on the brane is locally equivalent to Kerr it is not globally equivalent. This is because the identification we need to remove the conical singularities on the axis forces the solution to rotate at infinity. This can be seen as follows. Observe that when $\lambda = 0$ the function $G(x)$ is the same as in the absence of rotation. This means that the axis of φ lies along the same value of x_2 as the nonrotating AdS C-metric, and the periodicity (3.5) is determined only in terms of mA and not of a. However (5.8) shows that we must hold $\tilde{t} = t + ax_2^2\varphi$ (rather than $t + a\varphi$) fixed when we identify φ. If we set $t = \tilde{t} - ax_2^2\varphi$ into (5.11) (with $\lambda = 0$ and $k = -1$), then to leading order the metric is

$$ds^2 = \frac{1}{A^2}\left[-d\tilde{t}^2 - 2a(1 - x_2^2) d\tilde{t} d\varphi + d\rho^2 + \rho^2 d\varphi^2\right]. \tag{5.15}$$

We now want to compare this with the metric that describes a particle of spin J_3 in $2 + 1$ dimensions [17]

$$ds^2 = -(dt + 4G_3J_3 d\varphi)^2 + d\rho^2 + \rho^2 d\varphi^2 \tag{5.16}$$

(with φ periodically identified on fixed t surfaces). Clearly

$$J_3 = \frac{a(1 - x_2^2)}{4AG_3} \,. \tag{5.17}$$

For small mA, $x_2 \approx 1 - mA$, and we have seen that the four-dimensional mass and angular momentum of the Kerr black hole are $m = G_4M_4$ and $J_4 = M_4a/A$. Substituting into (5.17) and using (2.12) to relate the Newton's constants, we find

$$J_3 = J_4 \tag{5.18}$$

as well as $M_3 = M_4$. So even though the angular momentum in $2+1$ gravity is a global effect, it turns out to agree exactly with the four-dimensional angular momentum of the black hole. This is directly analogous to what we found for the mass in the absence of rotation [1], and can be understood by a similar argument. Before introducing the brane, the AdS C-metric describes a black hole accelerating in AdS. The cause of the acceleration is a cosmic string pulling on the black hole. When the string is attached to a spinning horizon, it is set into rotation. This rotation, like the conical deficit angle, is a global effect that can be detected on every section transverse to the axis. When the brane is introduced, the cosmic string is removed, but the boundary conditions on the brane essentially reproduce its effect. We expect that the agreement (5.18) continues to hold for large mA.

5.2 $\lambda > 0$: rotating BTZ black holes

When $\lambda > 0$, the metrics on the brane (5.11) are related to those of rotating solutions in AdS_3. In fact, if $mA = 0$ they are the same as BTZ black holes for $k = +1, 0$, and negative mass particles for $k = -1$. Thus, the mass and spin on the brane can be measured by comparing to the spinning BTZ solution [5],

$$ds^2 = -\left(\frac{\hat{\rho}^2}{\ell_3^2} - 8G_3M_3 + \frac{(4G_3J_3)^2}{\hat{\rho}^2}\right) d\hat{t}^2 + \left(\frac{\hat{\rho}^2}{\ell_3^2} - 8G_3M_3 + \frac{(4G_3J_3)^2}{\hat{\rho}^2}\right)^{-1} d\hat{\rho}^2 + \\ + \hat{\rho}^2 \left(d\hat{\varphi} - \frac{4G_3J_3}{\hat{\rho}^2} d\hat{t}\right)^2 \tag{5.19}$$

where $\hat{\varphi}$ has periodicity 2π (for fixed $\hat{t}$).

The most important feature introduced by a nonzero value of the rotation parameter a appears in the structure of $G(x)$. Notice that since $G(x)$ is now a quartic polynomial we are not guaranteed to always have a real root, even when $m \neq 0$. Consider for instance starting from a rotating BTZ black string ($k = +1$, $m = 0$), which extends throughout the entire spacetime. Now turn on a small, but nonzero m. When $a = 0$ we have seen that this results in the string being chopped off at one of its ends, the endpoint being marked by the smallest positive real root of $G(x)$. However, when $a \neq 0$ there is a range of (small) values of mA such that all roots of $G(x)$ are still complex, so the range of x is unrestricted, $-\infty < x < \infty$ (it becomes $0 \leq x < \infty$ once the brane is introduced). Therefore, for mA within this range of values, we still find a black string, which runs from the brane at $x = 0$ to the brane at $y = 0$.

As we let mA grow larger we reach a certain critical value at which $G(x)$ develops a positive double root x_2. Then the proper spatial distance to x_2, given as $\sim \int^{x_2} dx/\sqrt{G(x)}$, is infinite, so the black string has infinite proper length now.[10]

[10] The area remains finite, though.

The string extends infinitely far even if we put a second brane at $y = 0$ to compactify spacetime. The way this can happen is that this second brane also develops an infinite funnel centered at the double root $x = x_2$. The black string then extends down this funnel. Notice the period $\Delta\varphi$ can still be arbitrarily chosen.

For larger values of mA, the function $G(x)$ has real positive simple zeroes, and the upper value for x is the smallest of these zeroes, which is now a finite proper distance away. This marks the outermost extent of the black hole horizon inside the bulk. It is only in this case that the period $\Delta\varphi$, and hence the mass on the brane, is fixed (to the value (3.5)) once mA and a are fixed. Let us note that whether the horizon is extremal or not does not appear to influence the discussion of these issues.

These 'localized' black holes also produce a three-dimensional angular momentum, just as in the asymptotically flat case, $\lambda = 0$. The total angular momentum detected on the brane is obtained by comparing the asymptotic geometry to (5.19). As we saw earlier, a nonsingular rotation axis requires that the angular coordinate be identified periodically on surfaces of constant $\tilde{t}$ (5.8). So we change the time coordinate in (5.11) by $t = \tilde{t} - ax_2^2\varphi$ so that points now are identified as $(\tilde{t}, \rho, \varphi) \sim (\tilde{t}, \rho, \varphi + \Delta\varphi)$. It is easy to see that this results in the coefficient of the $d\tilde{t}d\varphi$ growing like ρ^2, instead of a constant, as in (5.19), which might appear to imply a diverging angular momentum. This, however, is an artifact of the coordinate system, and not a true global effect. To see this, notice that we can still define $\tilde{\varphi} = \varphi + f(\tilde{t}, \varphi)$ without changing the global identifications. In particular, letting $\varphi = \tilde{\varphi} - \frac{\lambda a x_2^2}{1-\lambda a^2 x_2^4}\tilde{t}$ we can remove the ρ^2 term in $d\tilde{t}d\tilde{\varphi}$, and finally find the asymptotic metric

$$A^2 ds^2 = -\frac{\lambda}{1-\lambda a^2 x_2^4}\rho^2 d\tilde{t}^2 + \rho^2(1-\lambda a^2 x_2^4)d\tilde{\varphi}^2 + \frac{4amAx_2^3}{1-\lambda a^2 x_2^4}d\tilde{t}d\tilde{\varphi} + \frac{d\rho^2}{\lambda\rho^2} + \cdots \tag{5.20}$$

The spin is then

$$J_3 = -\frac{amx_2^3}{2G_3}\left(\frac{\Delta\varphi}{2\pi}\right)^2 . \tag{5.21}$$

For a small range of parameters (corresponding to x_2 close to becoming a double root) $\tilde{t}$ becomes spacelike and $\tilde{\varphi}$ becomes timelike in (5.20). Since $\tilde{\varphi}$ is periodic, these solutions have closed timelike curves near infinity and are presumably unphysical.

6. Discussion

So far we have been considering black holes on branes with asymptotically flat or AdS geometries. It is easy to extend the discussion to the case of $\lambda < 0$, where the branes have compact volume and the geometry of deSitter universes. The analysis of the AdS C-metric in this case is quite straightforward, since apart from the different asymptotics at four-dimensional infinity, it is quite similar to the C-metric without a cosmological constant (which corresponds to $\lambda = -1$). Setting $k = -1$, it follows that for small mA, $H(y)$ has two negative roots.[11] The smaller root corresponds to a black hole horizon of finite size and the larger one is an acceleration horizon. The solution can be continued past the acceleration horizon, to find a second black hole accelerating in the opposite direction. When mA grows to a value such that $H(y)$ develops a double root, the black hole horizon and the acceleration horizon coincide. Limiting situations of this sort have been considered in [18]. For larger values of mA the singularity at $y = -\infty$ is naked.

On the brane, for small mA we find a black hole inside a dS_3 universe. Since there are no black holes in $2+1$ gravity with a positive cosmological constant, these are entirely a result of the modifications coming from the extra dimension. If the metric is continued past the three-dimensional cosmological horizon, then we recover a second black hole, which corresponds to the second black hole in AdS_4. The appearance of the naked singularity as mA grows corresponds to the black hole horizon on the brane becoming coincident with, and then larger than the cosmological horizon. We are not aware of any satisfactory definition of mass in dS_3, but we might expect (or define) the mass of black holes on the brane to be equal to the mass M_4 computed from thermodynamics in the bulk.

We have seen in section 4.2 that the solutions we have constructed do not appear to be the most general ones describing black holes on two-branes. In light of this, it is natural to wonder if the existence of a maximum possible mass of order the Planck scale for a localized black hole on the brane (discussed in section 3.2) is just an artifact of our family of solutions. While we do not know how to construct the most general solution, we now comment on possible alternative locations of the branes in the AdS C-metric. In contrast to the situation where no black holes are present (and the branes can be placed at any surface at constant x/y), when $m > 0$ the condition for the location of the branes — that the extrinsic curvature be proportional to the induced metric — is highly restrictive. Above, we found it was possible to place the two-branes at $x = 0$ and $y = 0$. More generally, for the metrics (2.1), the equation $K_{\mu\nu} \propto g_{\mu\nu}$ is solved on any surface $x = x_b$ such that $G'(x_b) = 0$ and $G(x_b) \neq 0$. Similarly, it is solved on any surface $y = y_b$ with $H'(y_b) = 0$ and $H(y_b) \neq 0$. However it turns out that the resulting solutions are equivalent to the ones already studied.

Let us consider the case of constant nonzero x in more detail. For $k = \pm 1$ it is possible to place a brane at the second root of $G'(x)$, $x_b = k/3mA$. However, one can always shift coordinates so that the brane is at $x = 0$ in the new coordinates. Define $\gamma = \sqrt{G(x_b)} = \sqrt{1 + k/27m^2A^2}$, and change

$$\begin{aligned} x &= x_b + \gamma\bar{x}\,, & y &= x_b + \gamma\bar{y}\,, \\ t &= \frac{\bar{t}}{\gamma}\,, & \varphi &= \frac{\bar{\varphi}}{\gamma}\,. \end{aligned} \tag{6.1}$$

[11] For simplicity we only discuss the non-rotating case.

To convert the resulting metric into one of the same form as (2.1) we just have to redefine the parameters as

$$A = \frac{\bar{A}}{\gamma}\,, \qquad k = -\bar{k}\,, \tag{6.2}$$

while leaving m unchanged. Notice $x = x_b$ is mapped onto $\bar{x} = 0$. One can also see that the sign of $\bar{\lambda} = -1 + 1/l^2\bar{A}^2$ may be different to that of λ. For illustration, consider placing a two-brane at $x_b = -1/3mA$ when $\lambda = 0$ and $k = -1$. In order for x_b to fall into the range of variation of x, we must require $mA > 1/3\sqrt{3}$ (this also ensures γ is real) which is in a black string regime. The transformation above results into $\bar{k} = +1$ and $\bar{\lambda} = 1/(27m^2A^2 - 1) > 0$. Conversely, if $k = +1$ and $\lambda = 1/27m^2A^2$, then a two-brane at $x = 1/3mA$ is asymptotically flat.

From our investigations in sections 3 and 4, we arrive at the following general picture: For $M_3 > 1/24G_3$, a gravitationally collapsed system can only form a BTZ black string. These black strings remain the stable configuration down to a certain transition mass, $M_3 \approx 1/32G_3$ for small λ, where the minimum transverse size of the black string shrinks to be of order the four-dimensional AdS radius. (Recall that our calculations of this transition are reliable only for $\lambda < 1$.) At around this mass, the black strings are destabilized by the Gregory-Laflamme instability [15], which causes them to break up and form a black hole localized on the Planck brane. In the small λ regime, one finds that at the transition point the localized black hole appears to have essentially the geometry of a large ($\rho_{\rm horz} \approx \lambda^{-1/2}$) BTZ black hole. That is, the $2mA/\rho$ corrections in eq. (3.6) are very small everywhere outside of the event horizon. In addition, at the transition point the horizon of these localized black holes extends out in the transverse space to come very close to the 'throat' at $r = \sqrt{\lambda}\ell_4$. Below the transition point down to $M_3 = 0$, black holes localized on the Planck brane are the stable end-point of a gravitational collapse. This continues to be true for $-1/8G_3 < M_3 < 0$, where these black holes are the only solutions with an event horizon. As M_3 approaches $-1/8G_3$, these black holes resemble four-dimensional Schwarzschild AdS black holes. At precisely $M_3 = -1/8G_3$, one is left with a slice of pure AdS_4 in the bulk spacetime on either side of the branes.

By considering the instability of the black string, one is led to a general argument for the sign of the correction terms to the gravitational potential on the brane. The following argument applies in all dimensions, but to be definite, we consider the standard case of a three brane in AdS_5. The usual four dimensional Schwarzschild black hole can be realized on the brane if the full solution is a black string [7]. When the black string is unstable, it caps off its ends to form a localized black hole. But this will always decrease the total horizon area unless the cross sectional area is increased at the same time. So if a localized black hole is going to be stable, it must have *larger* horizon area on the brane than the original Schwarzschild solution of the same mass. This tells us the sign of the corrections to the gravitational potential.

That is, if we have

$$g_{tt} = -1 + V(r) = -1 + \frac{2GM}{r} + \alpha \left(\frac{L}{r}\right)^{\beta} \frac{2GM}{r} \tag{6.3}$$

then we had better have $\alpha > 0$ in order that the horizon be at a larger radius than $r_H = 2GM$. This agrees with the sign obtained from a perturbative calculation in [2] (see also [13, 14]), and from our exact solutions in one lower dimension in section 3.

In light of the AdS/CFT correspondence, it has been suggested that the Randall-Sundrum scenario can be viewed as a coupling of the lower dimensional gravity on the brane to a strongly coupled conformal field theory which is dual to gravity in the bulk [19]. For the case of negative curvature branes, there is the intriguing possibility that we could apply this duality twice. Since the theory on the brane is $2+1$ AdS gravity (coupled to a $2+1$ CFT), it may be equivalent in some sense to a $1+1$ CFT! In a sense, one may have a holographic description of a holographic description of the theory.

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